

REAL DYNAMICS OF A FAMILY OF PLANE BIRATIONAL MAPS: TRAPPING REGIONS AND ENTROPY ZERO

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1. INTRODUCTION

We consider dynamics of the one parameter family of birational maps

$$(1) \quad f = f_a : (x, y) \mapsto \left(y \frac{x+a}{x-1}, x+a-1 \right).$$

This family was introduced and studied by Abarenkova, Anglès d'Auriac, Boukraa, Hassani, and Maillard, with results published in [A1–7]. We consider here real (as opposed to complex) dynamics, treating f_a as a self-map of \mathbf{R}^2 , and we restrict our attention to parameters $a > 1$. In order to discuss our main results, we let $\mathcal{B}^+, \mathcal{B}^- \subset \mathbf{R}^2$ be the sets of points with orbits diverging locally uniformly to infinity in forward/backward time, and we take $K \subset \mathbf{R}^2$ to be the set of points $p \in \mathbf{R}^2$ whose full orbits $(f^n(p))_{n \in \mathbf{Z}}$ are bounded.

In [BD1] we studied the dynamics of f_a for the parameter region $a < 0$, $a \neq -1$. In this case, \mathcal{B}^+ and \mathcal{B}^- are dense in \mathbf{R}^2 ; and the complement $\mathbf{R}^2 - \mathcal{B}^+ \cup \mathcal{B}^-$ is a non-compact set on which the action of f_a is very nearly hyperbolic and essentially conjugate to the golden mean subshift. In particular, f_a is topologically mixing on $\mathbf{R}^2 - \mathcal{B}^+ \cup \mathcal{B}^-$, and most points therein have unbounded orbits. The situation is quite different when $a > 1$.

Theorem 1.1. *If $a > 1$, then $\mathbf{R}^2 - \mathcal{B}^+ \cup \mathcal{B}^- = K$. Moreover, the set K is compact in \mathbf{R}^2 , contained in the square $[-a, 1] \times [-1, a]$.*

The sets \mathcal{B}^+ and K are illustrated for typical parameter values $a > 1$ in Figure 1. Note that all figures in this paper are drawn with \mathbf{R}^2 compactified as a torus $S^1 \times S^1 \cong (\mathbf{R} \cup \{\infty\}) \times (\mathbf{R} \cup \{\infty\})$. The plane is parametrized so that infinity is visible, with top/bottom and left/right sides identified with the two circles at infinity. All four corners of the square correspond to the point (∞, ∞) , which is a parabolic fixed point for f_a .

For each $a \neq -1$ there is a unique fixed point $p_{fix} = ((1-a)/2, (a-1)/2) \in \mathbf{R}^2$. For $a < 0$, p_{fix} is a saddle point, and for $a > 0$, p_{fix} is indifferent with $Df_a(p_{fix})$ conjugate to a rotation. For generic $a > 0$ this rotation is irrational, and f_a acts on a neighborhood of p_{fix} as an area-preserving twist map with non-zero twist parameter (see Proposition 5.1). As is shown in Figure 2, f_a exhibits KAM behavior for typical $a > 0$. In particular K has non-empty interior, and the restriction $f_a : K \rightarrow K$ is neither topologically mixing nor hyperbolic.

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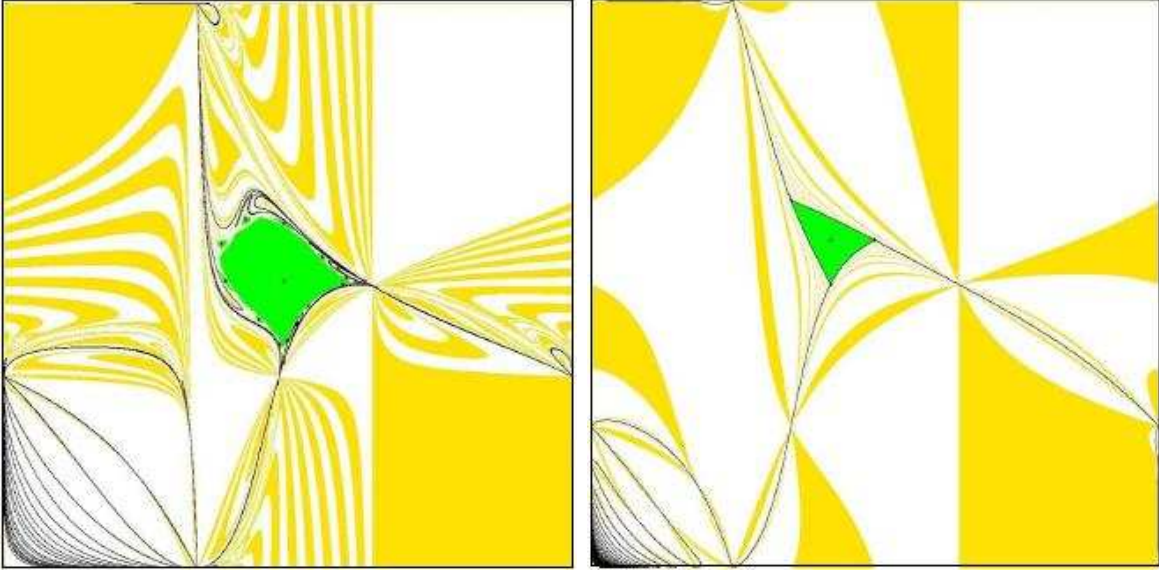


FIGURE 1. Dynamics of f for parameter values $a = 1.1$ (left) and $a = 2$ (right). Forward orbits of white points escape to infinity traveling up and to the right. Yellow points escape by alternating between the bottom right and upper left corners. The black curves are stable manifolds of a saddle three cycle (the vertices of the ‘triangle’). The green regions consist of points whose orbits are bounded in forward and backward time. Reflection about the line $y = -x$ corresponds to replacing f by f^{-1} . It leaves the green region invariant and exchanges stable and unstable manifolds.

It is known (see [DF] section 9) that f_a is integrable for the parameter values $a = -1, 0, \frac{1}{3}, \frac{1}{2}, 1$. In particular f_a has topological entropy zero for these parameters. On the basis of computer experiments, [A5, A7] conjectured that $a = 3$ is the (unique) other parameter where entropy vanishes. The map f_3 is not integrable, and in fact the complexified map $f_3 : \mathbf{C}^2 \circlearrowleft$ has entropy $\log \frac{1+\sqrt{5}}{2} > 0$ (see [BD1] and [Duj]). Nevertheless, we prove in Section 6 that not only does $f_3 : \mathbf{R}^2 \circlearrowleft$ have zero entropy; all points except p_{fix} in \mathbf{R}^2 are transient:

Theorem 1.2. *For $a = 3$, $K = \{p_{fix}\}$. The stable (and unstable) set of p_{fix} consists of three analytic curves. The three curves meet transversely at p_{fix} and have no other pairwise intersections in \mathbf{R}^2 .*

Thus, p_{fix} is the only periodic and the only non-wandering point in \mathbf{R}^2 . Figure 3 illustrates this theorem. As the figure makes evident, the stable arcs for p_{fix} intersect pairwise on a countable set at infinity. This reflects the fact that f is not a homeomorphism.

Often in this paper we will bound the number of intersections between two real curves by computing the intersection number of their complexifications. We will also use the special structure (see Figure 4) of f_a to help track the forward and backward images of curves. The versatility of these two techniques is seen from the fact that they apply in cases where maps

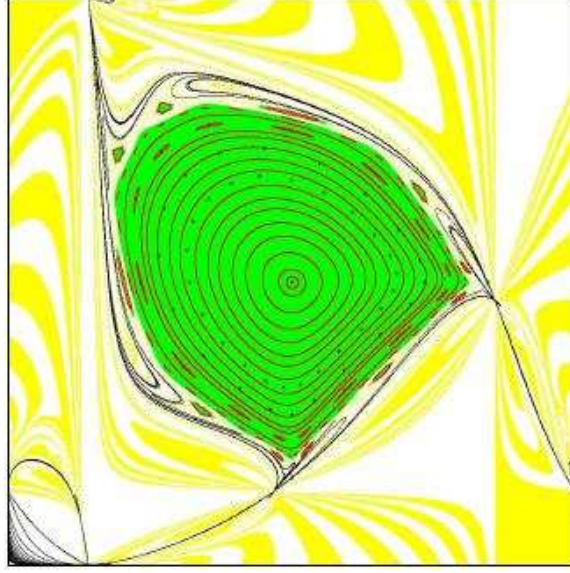


FIGURE 2. This is the left side of Figure 1 redrawn with coordinates changed to magnify the behavior near p_{fix} . Shown are several invariant circles about p_{fix} , as well as some intervening cycles and elliptic islands of large period.

have maximal entropy ([BD1, BD2]) and in the present situation where we will show that f_3 has zero entropy.

2. BACKGROUND

Here we recall some basic facts about the family of maps (1). Most of these are discussed at greater length in [BD1]. The maps (1) preserve the singular two form

$$\eta := \frac{dx \wedge dy}{y - x + 1}.$$

Each map $f = f_a$ is also reversible, which is to say equal to a composition of two involutions [BHM2, BHM1]. Specifically, $f = \tau \circ \sigma$

$$\tau(x, y) := \left(x \frac{a - y}{1 + y}, a - 1 - y \right), \quad \sigma(x, y) := (-y, -x).$$

In particular, $f^{-1} = \sigma \circ \tau$ is conjugate to f by either involution, a property that will allow us to infer much about f^{-1} directly from facts about f .

Though our goal is to understand the dynamics of f acting on \mathbf{R}^2 , it will be convenient to extend the domain of f first by complexification to all of \mathbf{C}^2 and then by compactifying each coordinate separately to $\mathbf{P}^1 \times \mathbf{P}^1$. This gives us a convenient way of keeping track of the complexity of algebraic curves in $\mathbf{P}^1 \times \mathbf{P}^1$. Any such curve V is given as the zero set of a rational function $R : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$. The *bidegree* $(j, k) \in \mathbf{N}^2$ of R , obtained by taking the degrees of R as a function of the first and second variables in turn, encodes the second homology class of V . In particular, if V and V' are curves with bidegrees (j, k) and (j', k')

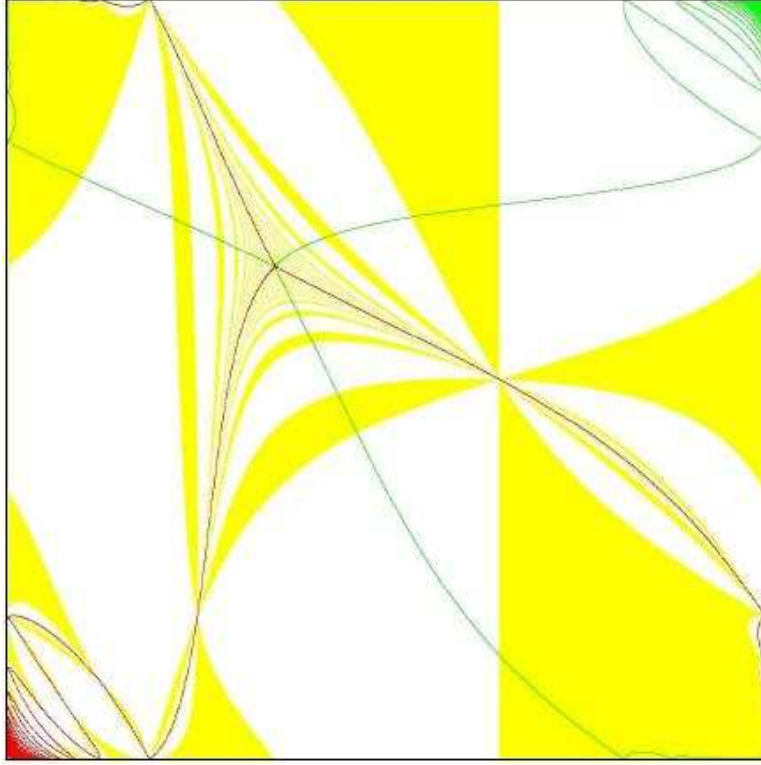


FIGURE 3. Dynamics of f when $a = 3$. The three cycle has now disappeared, and with it the twistmap dynamics. The indifferent fixed point is now parabolic, and its stable and unstable sets are shown in red and green, respectively.

having no irreducible components in common, then the number of (complex) intersections, counted with multiplicity, between V and V' , is

$$(2) \quad V \cdot V' = jk' + j'k.$$

We will use this fact at key points as a convenient upper bound for the number of *real* intersections between two algebraic curves.

Bidegrees transform linearly under our maps. Taking f^*V to be the zero set of $R \circ f$, we have

$$(3) \quad \text{bideg } f^*V = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{bideg } V.$$

Similarly, f_*V (the zero set of $R \circ f^{-1}$) has bidegree given by

$$(4) \quad \text{bideg } f_*V = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{bideg } V.$$

The divisor (η) of η , regarded as a meromorphic two form on $\mathbf{P}^1 \times \mathbf{P}^1$, is supported on the three lines $\{x = \infty\}$, $\{y = \infty\}$, $\{y = x - 1\}$ where η has simple poles. It follows from

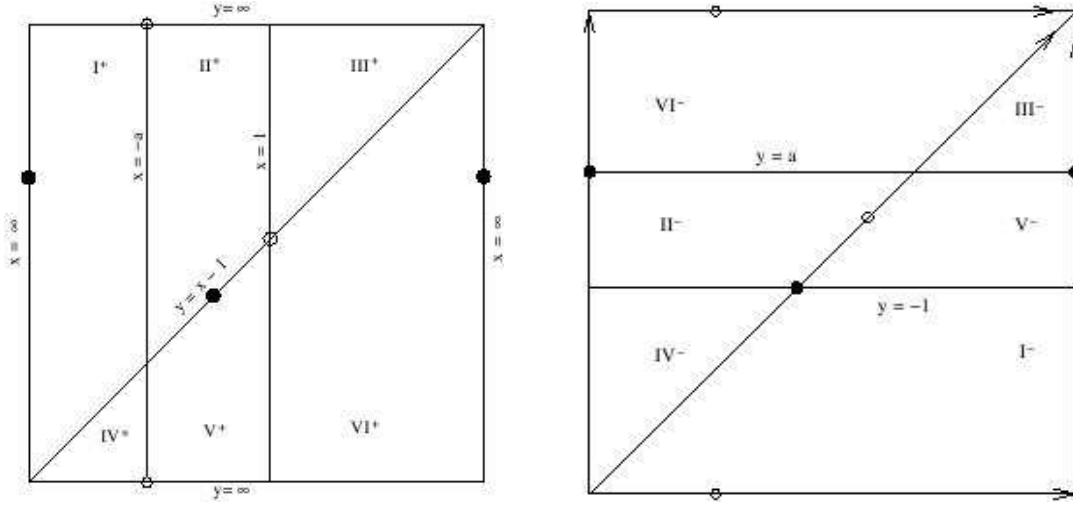


FIGURE 4. Partition of \mathbf{R}^2 by invariant curves and critical sets of f (left side) and f^{-1} (right side). Points of indeterminacy for f appear as hollow circles, whereas those for f^{-1} are shown as solid circles. Arrows on the right side indicate the direction in which f^2 translates points along $\text{supp}(\eta)$.

$f^*\eta = \eta$ that $\text{supp}(\eta)$ is invariant under f . Specifically, f interchanges the lines at infinity according to

$$(\infty, y) \mapsto (y, \infty) \mapsto (\infty, y + a - 1)$$

and maps $\{y = x - 1\}$ to itself by $(x, x - 1) \mapsto (x + a, x + a - 1)$. Thus f^2 acts by translation on each of the three lines separately. The directions of the translations divide parameter space into three intervals: $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$. In particular, the directions are the same for all $a \in (1, \infty)$, which is the range that concerns us here.

It should be stressed that $f : \mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow$ is *not* a diffeomorphism, nor indeed even continuous at all points. In particular, the critical set $C(f)$ consists of two lines $\{x = -a\}$ and $\{x = 1\}$, which are mapped by f to points $(0, 1)$ and (∞, a) , respectively. Applying the involution σ , one finds that $\{y = -1\}$ and $\{y = a\}$ are critical for f^{-1} , mapping backward to $(-a, \infty)$ and $(-1, 0)$, respectively. Clearly, f cannot be defined continuously at the latter two points. Hence we call each a *point of indeterminacy* and refer to $I(f) = \{(-a, \infty), (-1, 0)\}$ as the *indeterminacy set* of f . Likewise, $I(f^{-1}) = \{(0, 1), (\infty, a)\}$. For convenience, we let

$$I^\infty(f) := \bigcup_{n \in \mathbf{N}} I(f^n) = \bigcup_{n \in \mathbf{N}} f^{-n} I(f)$$

$$C^\infty(f) := \bigcup_{n \in \mathbf{N}} C(f^n) = \bigcup_{n \in \mathbf{N}} f^{-n} C(f)$$

denote the set of all points which are indeterminate/critical for high enough forward iterates of f . We point out that both $I^\infty(f)$ and $I^\infty(f^{-1})$ are contained in $\text{supp}(\eta)$. Since $I(f)$ is contained in $\text{supp}(\eta)$, it follows that $I^\infty(f)$ is a discrete subset of $\text{supp}(\eta)$ that accumulates only at (∞, ∞) . We also observe that for the parameter range $a > 1$, we have $I^\infty(f) \cap I^\infty(f^{-1}) = \emptyset$, a fact which will be useful to us below.

The closure $\overline{\mathbf{R}^2}$ of the real points in $\mathbf{P}^1 \times \mathbf{P}^1$ is just the torus $S^1 \times S^1$. The left side of Figure 4 shows how the critical set of f together with $\text{supp}(\eta)$ partition $\overline{\mathbf{R}^2}$ into six open sets, which we have labeled I^+ to VI^+ . The right side shows the partition by $\text{supp}(\eta)$ and the critical set of f^{-1} . As f is birational, each piece of the left partition maps diffeomorphically onto a piece of the right partition. We have chosen labels on the right side so that I^+ maps to I^- , etc.

3. TRAPPING REGIONS

For the rest of this paper we assume $a > 1$. Figure 4 is useful for determining images and preimages of real curves by f . In this section, we combine the information presented in the figure with intersection data gleaned from bidegrees to help identify two ‘trapping regions’ through which orbits are forced to wander to infinity. Our first trapping region is the set

$$T_0^+ := \{(x, y) \in \mathbf{R}^2 : x > 1, y > a\}$$

of points lying to the right of the critical set of f and above the critical set of f^{-1} .

Theorem 3.1. *The region T_0^+ is forward invariant by f . Forward orbits of points in T_0^+ tend uniformly to (∞, ∞) .*

Proof. Since $T_0^+ \subset III^+ \cup VI^+$ contains no critical or indeterminacy points of f , we have that $f(T_0^+)$ is a connected open subset. Since the left side of T_0^+ maps to a point, and $\partial T_0^+ \cap I(f) = \emptyset$, we see that in fact $f(T_0^+)$ is the region in $III^- \cup VI^-$ lying above and to the right of an arc $\gamma \subset f\{y = a\}$ that joins $(a, \infty) = f(\infty, a)$ to $(\infty, a) = f\{x = 1\}$.

By (4), the bidegree of $f\{y = a\} = f_*(y = a)$ is $(1, 1)$. Hence by (2), we see that $f\{y = a\}$ intersects $y = a$ in a single (a priori, possibly complex) point. Since (a, ∞) is one such intersection, we conclude that $\gamma \cap \{y = a\}$ contains no points in \mathbf{R}^2 . That is, γ lies above $\{y = a\}$. Similarly, $\gamma \cap \{x = 1\}$ contains at most one point. However, because $a > 1$, both endpoints of γ have x coordinates greater than 1, and the number of intersections between γ and $\{x = 1\}$ is therefore even. We conclude that $\gamma \cap \{x = 1\} = \emptyset$. This proves $f(T_0^+) \subset T_0^+$.

The same method further applies to show that points in T_0^+ have orbits tending uniformly to (∞, ∞) . If for $s, t > 0$, we let

$$R_{st} = \{(x, y) \in \mathbf{R}^2 : x > 1 + s, y > 1 + t\} \subset T_0^+,$$

then arguments identical to those above combined with the fact that $f\{x = 1 + s\} = \{y = a + s\}$ suffice to establish

$$f(R_{st}) \subset R_{t+a, s}.$$

Hence $f^2(R_{ss}) \subset R_{s+a-1, s+a-1}$. Since $a > 1$ and the closures $\overline{R_{ss}} \subset \overline{\mathbf{R}^2}$ decrease to the point (∞, ∞) as $s \rightarrow \infty$, the proof is finished. \square

Our second trapping region is more subtle. In particular, it has two connected components which are interchanged by f . The first component is

$$A := \{(x, y) \in \mathbf{R}^2 : x > 1, y < -x\}.$$

Lemma 3.2. *We have $A \cap f^{-1}(A) = \emptyset$ and $f(A) \subset f^{-1}(A)$.*

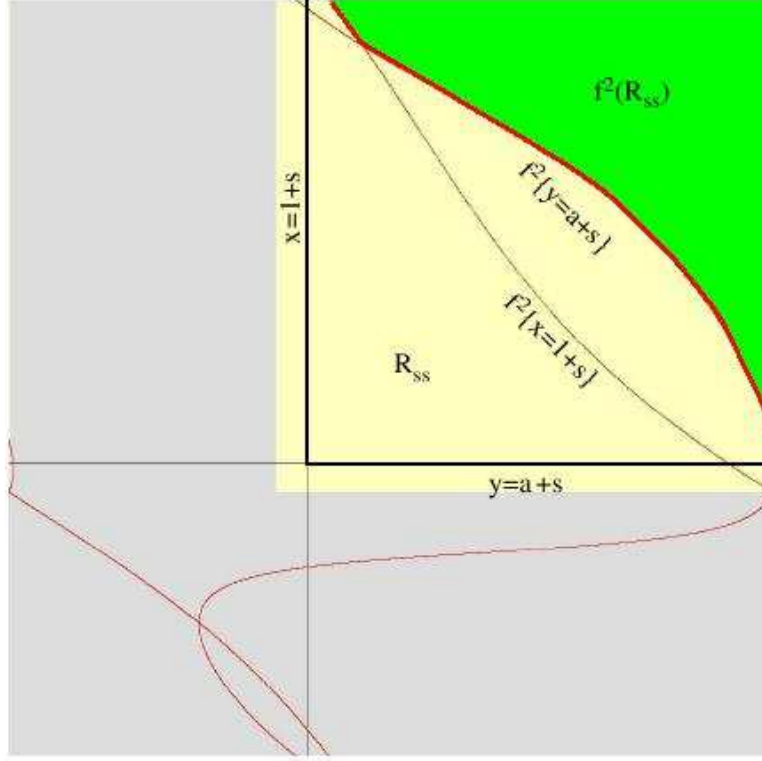


FIGURE 5. First trapping region (shaded yellow/green) for f . Also shown are the subregion R_{ss} (the square with lower left corner $(1+s, a+s)$ bounded by the thick black lines) and $f^2(R_{ss})$ (green region) for $s = .3$. The parameter value is $a = 1.2$

Proof. The set A lies entirely in region I^- shown on the right side of Figure 4, and therefore $f^{-1}(A)$ lies in region I^+ on the left side. In particular $A \cap f^{-1}(A) = \emptyset$.

To see that $f(A) \subset f^{-1}(A)$, we note that A is bounded in $\overline{\mathbf{R}^2}$ by portions of three lines, one of which $\{x = 1\}$ is critical for f . Moreover, \overline{A} contains no point in $I(f)$ or $I(f^{-1})$ and A itself avoids the critical sets of f and f^{-1} . Therefore, $f(A)$ is a connected open subset of \mathbf{R}^2 , bounded on the left by the segment $\{(-\infty, y) : y \geq a\}$ and the right by an arc γ in $f\{y = -x\}$ that joins $(-\infty, a)$ to $(-\infty, \infty)$. We claim that $f(A)$ lies below $\{y = -x\}$ and above the arc β in $f\{y = -1\} \cap VI^-$ that joins $(-\infty, a)$ to $(-a, \infty)$. Both these claims can be verified in the same fashion; we give the details only for $\{y = -x\}$.

Clearly $(-\infty, a) \in \gamma$ lies below $\{y = -x\}$, and $(-\infty, \infty) \in \gamma$ lies exactly on $\{y = -x\}$. Hence to establish that $f(A)$ is below $\{y = -x\}$ it suffices to show that γ does not intersect $\{y = -x\}$ at any other point. But the curve $f_*\{y = -x\}$ containing γ has bidegree $(1, 2)$ and $\{y = -x\}$ has bidegree $(1, 1)$, so there are a total of three complex intersections between the two curves. The intersection at $(-\infty, \infty)$ is actually a tangency, which accounts for two of the three intersections. The fact that $\{y = -x\}$ joins the critical line $\{x = -a\}$ to the

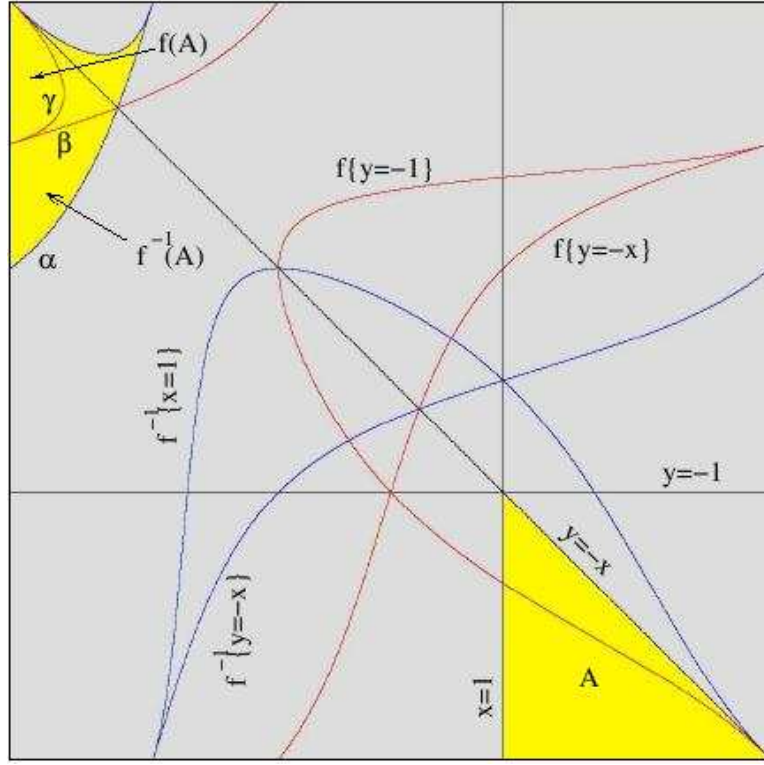


FIGURE 6. The second trapping region for (∞, ∞) , shown here in yellow together with the curves used in the proof of Lemma 6. Lines used in the proof are black, images of lines are red, and preimages are blue. The parameter value is $a = 3$.

critical line $\{x = 1\}$ through regions II^+ and V^+ implies that $f\{y = -x\}$ joins through an arc passing right from $(1, 0)$ to (∞, a) . This arc necessarily intersects $\{y = -x\}$ and accounts for the third intersection. Since there are no other intersections we conclude that γ is entirely below $\{y = -x\}$.

Having pinned down $f(A)$, we turn to $f^{-1}(A)$, which is bounded on the left by $\{(-\infty, y) : y \geq 1\}$, on the right by $f^{-1}\{x = 1\}$ and from above by $f^{-1}\{y = -x\}$. The top boundary component is in fact just $\sigma(\gamma)$ and therefore lies above $\{y = -x\}$. Therefore to show $f(A) \subset f^{-1}(A)$, it suffices to show that $f(A)$ lies above the portion α of $f^{-1}\{x = 1\}$ bounding $f^{-1}(A)$. For this, it suffices in turn to show simply that α lies below the arc β described above. This can also be accomplished with the method of the previous paragraph and we spare the reader the details. \square

We now define our second trapping region to be

$$T_1^+ := A \cup f^{-1}(A).$$

Theorem 3.3. *The region T_1^+ is forward invariant. Forward orbits of points in T_1^+ tend uniformly to (∞, ∞) , alternating between A and $f^{-1}(A)$.*

Proof. Forward invariance of T_1^+ follows immediately from Lemma 3.2. In order to show that points in T_1^+ have orbits tending uniformly to infinity, we consider diagonal lines

$$L_t := \{y = t(x - 1)\}, \quad L'_t := \{y = tx - 1\}$$

passing through the points $(1, 0) \in I(f)$ and $(0, -1) \in I(f^{-1})$, respectively.

Lemma 3.4. *The curve $f^2(L_t) \cap A$ (non-empty only for $t < -1$) lies strictly below L'_t .*

Proof. One computes easily that $f(L_t) = L'_{1/t}$ (we only include the *strict* transform of L_t in the image $f(L_t)$; the line $\{y = a\} = f(1, 0)$ appearing in f_*L_t is omitted). Because $Df^2_{(\infty, \infty)} = \text{id}$, it follows that $f^2(L_t) = f(L'_{1/t})$ is tangent to L'_t at (∞, ∞) . Moreover, $f^2(L_t) \cap II^- = f(L'_{1/t} \cap II^+)$ joins $(0, -1)$ to $(a, 0)$ and therefore intersects L'_t at $(0, -1)$. This gives us a total of three points (counting multiplicity) in $f^2(L_t) \cap L'_t - A$. On the other hand both L'_t and L_t have bidegree $(1, 1)$ regardless of t , so from (2) and (4), we find $L'_t \cdot f_*L'_{1/t} = 3$. It follows that

$$f^2(L_t) \cap L'_t \cap A = \emptyset.$$

Finally, since $f^2(L_t) \cap I^- = f(L'_{1/t} \cap I^+)$ joins $(0, -1)$ to (∞, ∞) , we conclude that it lies strictly below L'_t . \square

To complete the proof of Theorem 3.3, let $p \in T_1^+$ be given. We may assume in fact that $p \in A$. Then each point $p_n := (x_n, y_n) := f^{2n}(p)$, $n \geq 0$ lies in A . Therefore $p_n \in L_{t_n}$ for some $t_n < -1$. Because L'_t is below L_t , the previous lemma implies that (t_n) is a decreasing sequence. More precisely, $t_{n+1} < rt_n$, where $r = r(x_n) < 1$ increases to 1 as $x_n \rightarrow \infty$.

Now if (p_n) does not converge to (∞, ∞) , we have a subsequence (p_{n_j}) such that $x_{n_j} < M < \infty$. The previous paragraph implies that by further refining this subsequence, we may assume that $p_{n_j} \rightarrow (x, \infty)$ for some $x < \infty$. This, however, contradicts the facts that f is continuous on \overline{A} and that $f^n(x, \infty) \rightarrow (\infty, \infty)$. \square

Let us define the forward basin \mathcal{B}^+ of (∞, ∞) to be the set of points $p \in \overline{\mathbf{R}^2}$ for which there exists a neighborhood $U \ni p$ such that $f^n|_U$ is well-defined for all $n \in \mathbf{N}$ and converges uniformly to (∞, ∞) on U . Our definition of \mathcal{B}^+ here differs slightly from the one given in the introduction in that we now allow points at infinity. Note that $(\infty, \infty) \notin \mathcal{B}^+$, because $I^\infty(f)$ accumulates at (∞, ∞) .

Theorem 3.5. *The basin \mathcal{B}^+ is a forward and backward invariant, connected and open set that contains all points in $\text{supp}(\eta) \cup C^\infty(f) - I^\infty(f) - \{(\infty, \infty)\}$. Moreover, the following are equivalent for a point $p \in \overline{\mathbf{R}^2}$.*

- *The forward orbit of p is well-defined and unbounded.*
- *$p \in \mathcal{B}^+$.*
- *$f^n(p)$ is in the interior of $\overline{T_0^+ \cup T_1^+}$ for $n \in \mathbf{N}$ large enough.*

We remark that in this context, it might be more relevant to consider orbits that accumulate on $\text{supp}(\eta)$ rather than orbits which are unbounded. Theorem 3.5 holds regardless.

Proof. The basin \mathcal{B}^+ is open and invariant by definition. Since $I^\infty(f) \subset \text{supp}(\eta)$, we have that $\overline{I^\infty(f)}$ is discrete and accumulates only at (∞, ∞) . Therefore connectedness of \mathcal{B}^+ follows from invertibility of f .

Observe now that the $\overline{T_0^+ \cup T_1^+}$ contains a neighborhood of any point $(t, \infty), (\infty, t), (t, t-1) \in \text{supp}(\eta)$ for which t is large enough. From this it follows easily that $\text{supp}(\eta) - \overline{I^\infty(f)} \subset \mathcal{B}^+$. Since $C(f) - I(f)$ maps to $I(f^{-1})$ and $I^\infty(f^{-1}) \cap I^\infty(f) = \emptyset$, it further follows that $C^\infty(f) - I^\infty(f) \subset \mathcal{B}^+$.

The statements in the final assertion are listed from weakest to strongest, so it suffices to prove the first implies the last. So suppose that $p \in \overline{\mathbf{R}^2}$ is a point whose forward orbit is well-defined and unbounded. If $(f^n(p))_{n \geq 0}$ accumulates at $q \in \text{supp}(\eta) - I^\infty(f)$, then it also accumulates at every point in the forward orbit of q . We observed in the previous paragraph that $f^n(q)$ lies in the interior of $\overline{T_0^+ \cup T_1^+}$. Therefore since f^n is continuous on a neighborhood of q , it follows that $f^m(p) \in \mathcal{B}^+$ for some large m . By invariance, we conclude $p \in \mathcal{B}^+$, too.

The remaining possibility is that the forward orbit of p accumulates at infinity only at points in $I^\infty(f)$. This is impossible for the following reason. Since $I^\infty(f)$ is disjoint from $I^\infty(f^{-1})$, and since f and f^{-1} are conjugate via the automorphism σ , we have that the backward iterates (f^{-n}) converge uniformly to (∞, ∞) on a neighborhood $U \ni I^\infty(f)$. Thus points in U cannot recur to points in $I^\infty(f)$, and in particular, the forward orbit of p cannot accumulate on $I^\infty(f)$. \square

Because f and f^{-1} are conjugate via $(x, y) \mapsto (-y, -x)$, we immediately obtain trapping regions

$$T_0^- = \sigma(T_0^+), \quad T_1^- = \sigma(T_1^+)$$

for f^{-1} , for which the exact analogues of Theorems 3.1, 3.3, and 3.5 hold. In particular, the backward basin $\mathcal{B}^- = \sigma(\mathcal{B}^+)$ of (∞, ∞) includes all points in $C^\infty(f)$ and all points in $\text{supp}(\eta) - I^\infty(f^{-1}) - \{(\infty, \infty)\}$. Using once again the fact that $I^\infty(f)$ is disjoint from $I^\infty(f^{-1})$, we have

Corollary 3.6. *All points except (∞, ∞) with unbounded forward or backward orbits are wandering. Such points include all of $\text{supp}(\eta)$, $I^\infty(f)$, $I^\infty(f^{-1})$, $C^\infty(f)$ and $C^\infty(f^{-1})$*

4. POINTS WITH BOUNDED ORBITS

In this section we turn our attention to the set

$$K = \{p \in \mathbf{R}^2 : (f^n(p))_{n \in \mathbf{Z}} \text{ is bounded}\}$$

of points with bounded forward and backward orbits. We begin by emphasizing an immediate implication of Theorem 3.5.

Corollary 4.1. *K is a compact, totally invariant subset of $\mathbf{R}^2 - \text{supp}(\eta)$ containing no critical or indeterminate point for any iterate of f . Any non-wandering point in $\overline{\mathbf{R}^2}$ except (∞, ∞) belongs to K .*

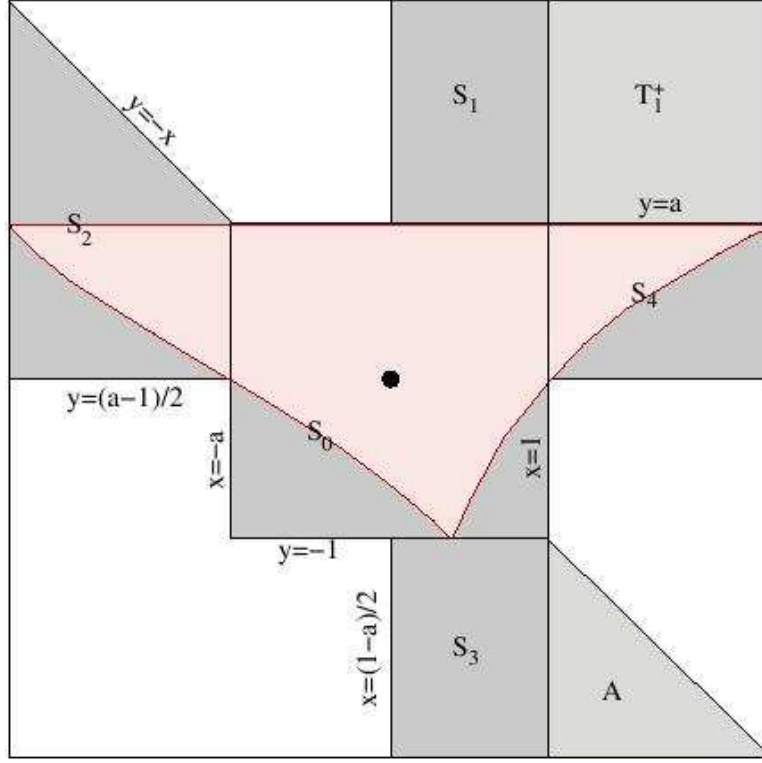


FIGURE 7. The regions S_0 through S_4 are quadrilaterals. The dot in the center is the fixed point, and the pink shaded region is $f(S_0)$. The regions T_0^+ and A are also shown. Observe that if all shaded regions are reflected about the line $y = -x$ (i.e. by the involution σ , then S_0 is sent to itself and the images of the remaining regions exactly fill the complementary white regions.

To study K we define several auxiliary subsets of \mathbf{R}^2 . Letting $(x_0, y_0) := (\frac{1-a}{2}, \frac{a-1}{2})$ denote the coordinates of the unique finite fixed point for f , we set

$$\begin{aligned} S_0 &:= [-a, 1] \times [-1, a], & S_1 &:= [x_0, 1] \times [a, \infty], \\ S_2 &:= \{(x, y) : x \leq -1, \frac{1}{2}(a-1) \leq y \leq -x\}, & S_3 &:= [x_0, 1] \times [-\infty, -1], \\ S_4 &:= [1, \infty] \times [y_0, a]. \end{aligned}$$

These sets are shown in Figure 7 with the image $f(S_0)$ superimposed. Observe that S_0 is just the square in \mathbf{R}^2 cut out by the critical sets of f and f^{-1} , and that the other four regions, three rectangles and a trapezoid (S_2), are arranged around S somewhat like ‘blades’ on a fan. Theorem 4.2 shows that f ‘rotates’ the blades counterclockwise. It also shows that S_0 acts as a kind of reverse trapping region for K since a point in S_0 whose orbit leaves S_0 cannot return again.

Theorem 4.2. *The sets S_0, \dots, S_4 satisfy the following:*

- (i) $T_0^+, T_1^+, S_0, \dots, S_4$, together with their images under the involution σ , cover \mathbf{R}^2 ;
- (ii) $f(S_0) \subset S_0 \cup S_2 \cup S_4$;
- (iii) $f(S_1) \subset S_2$;

- (iv) $f(S_2) \subset S_3 \cup A$;
- (v) $f(S_3) \subset S_4$;
- (vi) $f(S_4) \subset S_1 \cup T_0^+$;
- (vii) any point $p \in S_1 \cup \dots \cup S_4$ has a forward orbit tending to (∞, ∞) .

From this theorem, we have immediately that

Corollary 4.3. *For every point $p \in \mathbf{R}^2$, we exactly one of the following is true.*

- $f^n(p) \notin S_0$ for all $n \in \mathbf{N}$.
- There exists $N \in \mathbf{N}$ such that $f^n(p) \notin S_0$ for $n < N$ and $f^n(p) \in K$ for all $n \geq N$.
- There exist $N \leq M \in \mathbf{N}$ such that $f^n(p) \in S_0$ for all $N \leq n \leq M$ and $f^n(p) \notin S_0$ otherwise.

In the first two cases, p has a forward orbit tending to (∞, ∞) through T_0^+ or T_1^+ .

Since f and f^{-1} are conjugate by σ , we further have

Corollary 4.4. $K \subset S_0$.

Theorem 3.5 and Corollary 4.4 combine to yield Theorem 1.1 in the introduction. We spend the rest of this section proving Theorem 4.2. The first assertion is obvious from Figure 7.

Proof of Assertion (ii). Since S_0 lies between the critical lines of f in regions II^+ and V^+ (see Figure 4), $f(S_0)$ is the closure of a connected open set in regions II^- and V^- . Moreover, ∂S_0 contains the point $(1, 0) \in I(f)$, which maps to $y = a$. Hence $y = a$ constitutes the upper boundary of S_0 .

The vertical sides of S_0 are critical for f and map to points, so S_0 is bounded below by the intersections α and β of $f\{y = a\}$ and $f\{y = -1\}$, respectively, with $II^- \cup V^-$. From Figure 4 one finds that α stretches right from $(-\infty, a)$ to $(0, -1)$ and that β stretches left from $(0, -1)$ to (∞, a) . Moreover, one computes from 4 and 2 that each of these curves intersects each horizontal line no more than once. Hence α has non-positive slope at all points and β has non-negative slope. Since $(-a, y_0) = f(x_0, a) \in \alpha$ and $(1, y_0) = f(x_0, -1) \in \beta$, it follows that $f(S_0) - S_0$ is contained in S_2 and S_4 . \square

To verify the remaining assertions, it is useful to choose non-negative ‘adapted’ coordinates on S_1 through S_4 as follows. Each boundary ∂S_j , $1 \leq j \leq 4$, contains a segment ℓ of either the horizontal or vertical line passing through the fixed point (x_0, y_0) . We choose coordinates $(x_j, y_j) = \psi_j(x, y)$ on S_j so that $\ell \cap S_0$ becomes the origin (marked in Figure 7, distance to ℓ becomes the x_j coordinate and distance along ℓ to $\ell \cap S_0$ becomes the y_j coordinate. Thus, for example, we obtain coordinates $(x_1, y_1) = \psi_1(x, y) := (x - x_0, y - a)$ on S_1 and coordinates $(x_2, y_2) = \psi_2(x, y) := (y - y_0, 1 - x)$ on S_2 , etc.

Proof of assertions (iii) to (vi). To establish assertion (iii), we let $(x_1, y_1) = \psi_1(x, y)$ be the adapted coordinates of a point $(x, y) \in S_1$ and $(x_2, y_2) = \psi_2 \circ f(x, y)$ the coordinates of its image. Then $0 \leq x_1 \leq \frac{a+1}{2}$ and $0 \leq y_1$, and direct computation gives that

$$(x_2, y_2) := \psi_2 \circ f \circ \psi_1^{-1}(x_1, y_1) = \left(x_1, \frac{4ax_1 + y_1 + ay_1 + 2x_1y_1}{1 + a - 2x_1} \right).$$

Hence (x_2, y_2) satisfy the same inequalities as (x_1, y_1) , and it follows that $f(x, y) \in S_2$.

Verifying assertion (iii) is more or less the same, though messier because of S_2 is not a rectangle. If $(x, y) \in S_2$, then $(x_2, y_2) := \psi(x, y)$ satisfies $0 \leq x_2 \leq y_2 + \frac{a+1}{2}$ and $0 \leq y_2$. One computes

$$(x_3, y_3) := \psi_3 \circ f(x, y) = \left(\frac{-1 + a^2 + 2y_2(x_2 + a - 1)}{2(y_2 + a + 1)}, y_2 \right).$$

Since $a > 1$, we see that $x_3, y_3 \geq 0$. So to complete the proof, it suffices to show that $x_3 \leq y_3 + \frac{a+1}{2}$:

$$y_3 + \frac{a+1}{2} - x_3 = \frac{2 + 2a + y_2(5 + a + 2(y_2 - x_2))}{2(y_2 + a + 1)} > \frac{y_2(5 + a + (a+1))}{2(y_2 + a + 1)} > 0.$$

We leave it to the reader to verify assertions (v) and (vi). \square

Lemma 4.5. *Let $p \in S_1 \cup S_2 \cup S_3 \cup S_4$ be any point. Then the adapted x -coordinate of $f^2(p)$ is positive. If, moreover, the adapted y -coordinate of p is positive and $f^2(p) \in S_1 \cup S_2 \cup S_3 \cup S_4$, then the adapted y -coordinate of $f(p)$ is larger than that of p .*

Proof. In the same way we proved assertions (iii) through (vi) above, one may verify that the restriction of f to $S_1 \cup S_3$ preserves the adapted x -coordinate of a point, and the images of S_2 and S_4 do not contain points with adapted x -coordinate equal to zero. Hence $f^2(p)$ cannot have adapted x -coordinate equal to zero.

Similarly, one finds that the restriction of f to S_2 and S_4 preserves adapted y -coordinates, whereas f increases the adapted y -coordinates of points in S_1 and S_3 . For example, if $p = (x_1, y_1) \in S_1$, we computed previously that $f(p)$ has adapted y -coordinate

$$y_2 = \frac{4ax_1 + y_1 + ay_1 + 2x_1y_1}{1 + a - 2x_1} \geq y_1 + \frac{4ax_1}{1 + a - 2x_1} \geq y_1$$

with equality throughout if and only if $x_1 = 0$. \square

Proof of assertion (vii). Suppose the assertion is false. Then by assertions (iii) through (vi) it follows that there is a point p whose forward orbit is entirely contained $S_1 \cup S_2 \cup S_3 \cup S_4$. Moreover, by Theorem 3.5 the forward orbit of p must be bounded. Therefore, we can choose $q \in S_1 \cup S_2 \cup S_3 \cup S_4$ to be an accumulation point of $(f^n(p))_{n \in \mathbf{N}}$ whose adapted y -coordinate is as large as possible.

So on the one hand $f^2(q)$ is an accumulation point of the forward orbit of p and cannot, by definition of q , have adapted y -coordinate larger than that of q . But on the other hand, the first assertion of Lemma 4.5 implies that q cannot have adapted x -coordinate equal to zero, and therefore the second assertion tells us that $f^2(q)$ must have larger adapted y -coordinate than q . This contradiction completes the proof. \square

5. BEHAVIOR NEAR THE FIXED POINT

When $a \geq 0$, the eigenvalues of $Df(p_{fix})$ are a complex conjugate pair $\lambda, \bar{\lambda}$ of modulus one, with

$$\lambda = e^{i\gamma_0} = \frac{i + \sqrt{a}}{i - \sqrt{a}}.$$

Hence $Df(p_{fix})$ is conjugate to rotation by an angle γ_0 that decreases from 0 to $-\pi$ as a increases from 0 to ∞ . When $\gamma_0 \notin \mathbf{Q}\pi$ is an irrational angle, it is classical that f can be put formally into Birkhoff normal form:

Proposition 5.1. *If λ is not a root of unity, there is a formal change of coordinate $z = x + iy$ in which $p_{fix} = 0$, and f becomes*

$$z \mapsto z \cdot e^{i(\gamma_0 + \gamma_2|z|^2 + \dots)}$$

where γ_0 is as above, and

$$\gamma_2(a) = \frac{4(3a - 1)}{\sqrt{a}(a - 3)(1 + a)^2}.$$

Hence the ‘twist’ parameter γ_2 is well-defined and non-zero everywhere except $a = \frac{1}{3}$ and $a = 3$, changing signs as it passes through these two parameters.

For the proof of the proposition, we briefly explain how the Birkhoff normal form and attendant change of coordinate are computed (see [SM], §23 for a more complete explanation). We start with a matrix C whose columns are complex conjugates of each other and which satisfies $C^{-1}Df(p_{fix})C = \text{diag}(\lambda, \bar{\lambda})$. Conjugating $C^{-1} \circ f_a \circ C$, we “complexify” the map f_a , using new variables $(s, t) = (z, \bar{z})$, in which the map becomes

$$f : s \mapsto \lambda s + p(s, t), \quad t \mapsto \bar{\lambda} t + q(s, t),$$

and the power series coefficients of q are the complex conjugates of the coefficients of p . We look for a coordinate change $s = \phi(\xi, \eta) = \xi + \dots$, $t = \psi(\xi, \eta) = \eta + \dots$, such that the coefficients of ϕ and ψ are complex conjugates, and which satisfies $u \cdot \phi = p(\phi, \psi)$, for some $u(\xi, \eta) = \alpha_0 + \alpha_2\xi\eta + \alpha_4\xi^2\eta^2 + \dots$. Solving for the coefficients, we find $u = \exp(i(\gamma_0 + \gamma_2\xi\eta + \dots))$, with γ_0 and γ_2 as above. Returning to the variables $s = z$ and $t = \bar{z}$ gives the normal form.

For the rest of this section, we restrict our attention to the case $a = 3$. As was noted in [BHM2], this is the parameter value for which a 3-cycle of saddle points coalesces with p_{fix} . When $a = 3$, we have $\gamma_0 = -2\pi/3$. Translating coordinates so that p_{fix} becomes the origin, we have

$$\begin{aligned} f^3(x, y) &= (x, y) + Q + O(|(x, y)|^3) \\ Q = (Q_1, Q_2) &= (x^2/2 + xy + y^2, x^2 + xy + y^2/2). \end{aligned}$$

Let us recall a general result of Hakim [Hak] on the local structure of a holomorphic map which is tangent to the identity at a fixed point. A vector v is said to be characteristic if Qv is a multiple of v . The characteristic vectors v for the map f_a are $(1, -\frac{1}{2})$, $(1, 1)$, and $(1, 2)$. Hakim [Hak] shows that for each characteristic v with $Qv \neq 0$, there is a holomorphic embedding $\varphi_v : \Delta \rightarrow \mathbf{C}^2$ which extends continuously to $\bar{\Delta}$ and such that $\varphi_v(1) = (0, 0)$, and the disk $\varphi_v(\Delta)$ is tangent to v at $(0, 0)$. Further, $f(\varphi_v(\Delta)) \subset \varphi_v(\Delta)$, and for every $z \in \varphi_v(\Delta)$, $\lim_{n \rightarrow \infty} f^n z = (0, 0)$. The disk $\varphi_v(\Delta)$ is a \mathbf{C}^2 analogue of an ‘attracting petal’ at the origin. Applied to our real function f_a , this means that for each of the three vectors v , there is a real analytic “stable arc” $\gamma_v^s \subset \mathbf{R}^2$, ending at $(0, 0)$ and tangent to v . Considering f^{-1} , we have an “unstable arc” γ_v^u approaching $(0, 0)$ tangent to $-v$. These two arcs fit

together to make a C^1 curve, but they are not in general analytic continuations of each other.

Let us set $r(1, y) = Q_2(1, y)/Q_1(1, y)$ and write $v = (1, \eta)$. Then $a(v) := r'(\eta)/Q_1(1, \eta)$ is an invariant of the map at the fixed point. If we make a linear change of coordinates so that the characteristic vector $v = (1, 0)$ points in the direction of the x -axis, then we may rewrite f in local coordinates as

$$(x, u) \mapsto (x - x^2 + O(|u|x^2, x^3), u(1 - ax) + O(|u|^2x, |u|x^2)), \quad (6)$$

(see [Hak]). For the function (5), we find that $a(v) = -3$ for all three characteristic vectors. We conclude that the stable arcs γ_v^s are weakly repelling in the normal direction.

Another approach, which was carried out in [A7], is to find the formal power series expansion of a uniformization of γ_v^s at p_{fix} .

6. THE CASE $a = 3$

In this section we continue with the assumption $a = 3$, our aim being to give a global treatment of the stable arcs for p_{fix} . The basic idea here is that the behavior of $f|_{S_0}$ is controlled by invariant cone fields on a small punctured neighborhood of p_{fix} .

In order to proceed, we fix some notation. Let $\phi(x, y) = \phi_0(x, y) = x + 1$ and for each $j \in \mathbf{Z}$, let $\phi_j(x, y) = \phi \circ f^{-j}(x, y)$. Then $\phi_{j+k} = \phi_j \circ f^{-k}$ and in particular, $\phi_j(p_{fix}) = \phi_0 \circ f^{-j}(p_{fix}) = \phi_0(p_{fix}) = 0$ for every $j \in \mathbf{Z}$. We will be particularly concerned with the cases $j = -1, 0, 1, 2$ and observe for now that the level set $\{\phi_j = s\}$ is

- a horizontal line $\{x = -1 + s\}$ when $j = 0$;
- a vertical line $\{y = 1 + s\}$ when $j = 1$;
- a hyperbola with asymptotes $\{x = -3\}$, $\{y = -1 + s\}$ when $j = -1$; and
- a hyperbola with asymptotes $\{x = 1 - s\}$, $\{y = 3\}$ when $j = 2$.

Most of our analysis will turn on the interaction between level sets of ϕ_{-1} and ϕ_3 .

Proposition 6.1. *$\{\phi_2 = 0\}$ is tangent to $\{\phi_{-1} = 0\}$ at p_{fix} . Moreover $S_0 \cap \{\phi_2 > 0\} \subset S_0 \cap \{\phi_{-1} > 0\}$.*

Proof. The first assertion is a consequence of the facts that $f^3\{\phi_{-1} = 0\} = \{\phi_2 = 0\}$ and that Df^3 is the identity at p_{fix} . Since both curves in question are hyperbolas with horizontal and vertical asymptotes, it follows that p_{fix} is the *only* point where the curves meet.

The asymptotes of $\{\phi_{-1} = 0\}$ are the bottom and left sides of S_0 , whereas those of $\{\phi_2 = 0\}$ are the top and right sides. Therefore the zero level set of ϕ_{-1} intersects S_0 in a connected, concave up arc; and the zero level set of ϕ_2 meets S_0 in a connected concave down arc. It follows that the first level set is above and to the right of the second. Finally, direct computation shows that ϕ_{-1} and ϕ_2 are positive at the lower left corner of S_0 . This proves the second assertion in the proposition. \square

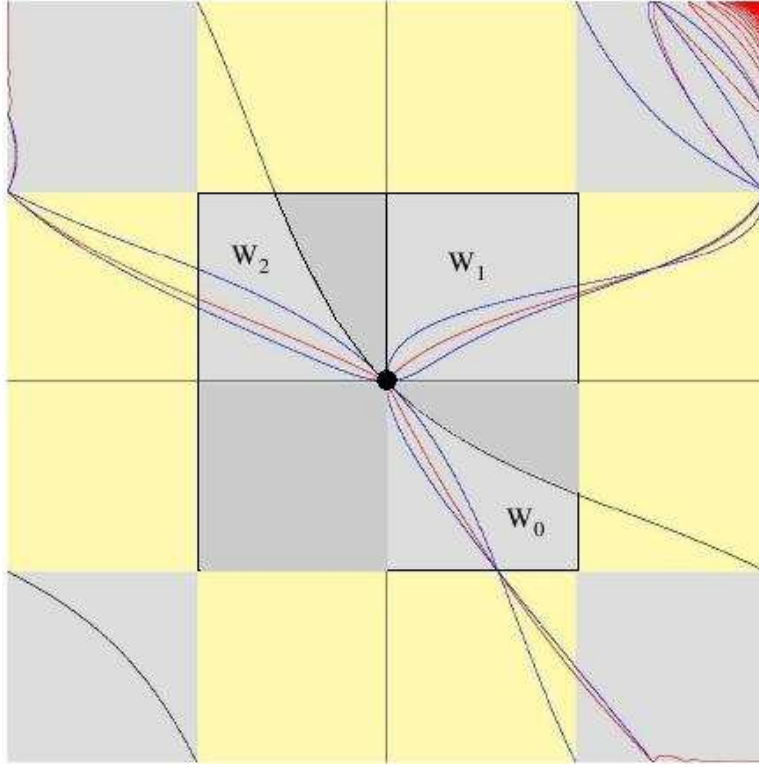


FIGURE 8. Unstable wedges. The square in the center is S_0 , with the fixed point in the middle. Boundaries of the wedges are the black curves. The images of the wedges under f^3 are bounded by the blue curves. The unstable curves for the fixed point are red.

Using the functions ϕ_j , $j = -1, 0, 1$, we define ‘unstable wedges’ $W_j \subset S_0$ emanating from p_{fix} :

$$\begin{aligned} W_0 &= \{(x, y) \in S_0 : 0 \leq \phi_{-1}, \phi_0\} \\ W_1 &= \{(x, y) \in S_0 : 0 \leq \phi_0, \phi_1\} \\ W_2 &= \{(x, y) \in S_0 : 0 \leq \phi_{-1}, \phi_1\} \end{aligned}$$

Proposition 6.2. *We have $f(W_0) \cap S_0 \subset W_1$, $f(W_1) \cap S_0 \subset W_2$, and $f(W_2) \cap S_0 \subset W_0$.*

Proof. It is immediate from definitions that $f(W_0) \cap S_0 \subset \{\phi_0, \phi_1 \geq 0\} \cap S_0 = W_1$.

Likewise,

$$f(W_1) \cap S_0 \subset \{\phi_1, \phi_2 \geq 0\} \cap S_0 \subset \{\phi_1, \phi_{-1} \geq 0\} \cap S_0 = W_2.$$

The second inclusion follows from the Proposition 6.1. Similar reasoning shows that $f(W_2) \cap S_0 \subset W_0$. \square

Lemma 6.3. *For each neighborhood U of p_{fix} , there exists $m = m(U) > 0$ such that*

$$\phi_{-1}(p) + m \leq \phi_2(p) \leq -m.$$

for every $p \in W_0 - U$.

Proof. Given $p \in W_1$, we write $p = (-1 + s, 1 + t)$ where $0 \leq s, t \leq 2$ and compute

$$\phi_2(p) = \frac{-2s - 2t - st}{t + 2} = \frac{s(-2 + t/2) + t(-2 + s/2)}{t + 2} < -\frac{s + t}{t + 2} \leq -\frac{s + t}{4};$$

and

$$\phi_2(p) - \phi_{-1}(p) = \frac{2(s^2 + t^2) + st(8 + s - t)}{(2 - s)(t + 2)} \geq \frac{2(s^2 + t^2)}{4}$$

Since the quantities $s^2 + t^2$ and $s + t$ are both bounded below by positive constants on $W_0 - U$, the lemma follows. \square

Proposition 6.4. *Let U be any neighborhood of p_{fix} . Then there exists $N > 0$ such that $f^n(p) \notin S_0$ for every $p \in W_0 \cup W_1 \cup W_2 - U$ and every $n \geq N$.*

Proof. By Proposition 6.2, we can assume that the point p in the statement of the lemma lies in $W_0 - U$. Since ϕ_2 is continuous on W_0 with $\phi_2(p_{fix}) = 0$ and $\phi_2 < 0$ elsewhere on W_0 , we may assume that U is of the form $\{\phi_2 > -\epsilon\}$ for some $\epsilon > 0$.

Suppose for the moment that $f^3(p) \in S_0$. Then Proposition 6.2 implies that $f^3(p) \in W_0$. By Lemma 6.3, we then have

$$\phi_2 \circ f^3(p) - \phi_2(p) = \phi_{-1}(p) - \phi_2(p) < -m(U).$$

In particular, $f^3(p) \in W_0 - U$. Repeating this reasoning, we find that if $p \in W_0 - U$ and $f^{3j}(p) \in S_0$ for $j = 1, \dots, J$, then

$$\phi_2 \circ f^{3J}(p) < -Jm(U).$$

On the other hand ϕ_2 is bounded below on W_0 (e.g. by -4). So if $J > 4/m(U)$, it follows that $f^{3j}(p) \notin S_0$ for some $j \leq J$. By Corollary 4.3, we conclude that $f^n(p) \notin S_0$ for all $n \geq 3j$. \square

Let us define the local stable set of p_{fix} to be

$$W_{loc}^s(p_{fix}) := \{p \in \mathbf{R}^2 : f^n(p) \in S_0 \text{ for all } n \in \mathbf{N} \text{ and } f^n(p) \rightarrow p_{fix}\},$$

with the local unstable set $W_{loc}^u(p_{fix})$ defined analogously.

Corollary 6.5. *The wedges W_0, W_1, W_2 meet $W_{loc}^s(p_{fix})$ only at p_{fix} , whereas they entirely contain $W_{loc}^u(p_{fix})$. Thus*

$$(5) \quad W_{loc}^s(p_{fix}) = W^s(p_{fix}) \cap S_0 = \{p \in \mathbf{R}^2 : f^n(p) \in S_0 \text{ for all } n \in \mathbf{N}\},$$

and similarly for W_{loc}^u .

Proof. The first assertion about $W_{loc}^s(p_{fix})$ follows from Propositions 6.2 and 6.4. Reversibility of f then implies the first assertion about $W_{loc}^u(p_{fix})$ is disjoint from the sets $\sigma(W_0), \sigma(W_1), \sigma(W_2)$.

We will complete the proof of the first assertion about W_0

$$S_0 \subset \bigcup_{j=0}^2 W_j \cup \sigma(W_j).$$

Observe that $\phi_0 \circ \sigma = -\phi_1$. Hence,

$$\phi_{-1} \circ \sigma = \phi_0 \circ f \circ \sigma = -\phi_1 \circ f^{-1} = -\phi_2.$$

Thus $\sigma(W_0) = \{\phi_1, \phi_2 \leq 0\}$, etc. The desired inclusion is therefore an immediate consequence of Proposition 6.1.

The first equality in (5) follows from Corollary 4.3, and the second from the analogue of Proposition 6.4 for f^{-1} . \square

Each of the wedges W_j admits an ‘obvious’ coordinate system identifying a neighborhood of p_{fix} in W_j with a neighborhood of $(0, 0)$ in $\mathbf{R}_+^2 := \{x, y \geq 0\}$. Namely, we let $\Psi_0 : W_0 \rightarrow \mathbf{R}^2$ be given by $\Psi_0 = (\phi_{-1}, \phi_0)$, $\Psi_1 : W_1 \rightarrow \mathbf{R}^2$ by $\Psi_1 = (\phi_0, \phi_1)$, and $\Psi_2 : W_2 \rightarrow \mathbf{R}^2$ by $\Psi_2 = (\phi_1, \phi_{-1})$. If $U \subset \mathbf{R}_+^2$ is a sufficiently small neighborhood of $(0, 0)$, then in light of Proposition 6.2, the maps $f_{01}, f_{12}, f_{20} : U \rightarrow \mathbf{R}_+^2$ given by $f_{ij} = \Psi_j \circ f \circ \Psi_i^{-1}$ are all well-defined. In fact, f_{01} is just the identity map. The other two are more complicated, but in any case a straightforward computation shows

Proposition 6.6. *If $x, y \geq 0$ are sufficiently small, the entries of the matrix $Df_{ij}(x, y)$ are non-negative.*

The proposition tells us that there are forward invariant cone fields defined near p_{fix} on each of the W_j ; i.e. the cones consisting of vectors with non-negative entries in the coordinate system Ψ_j . In this spirit, we will call a connected arc $\gamma : [0, 1] \rightarrow W_j \cap \overline{U}$ *admissible* if $\gamma(0) = p_{fix}$, $\gamma(1) \in \partial U$ and both coordinates of the function $\Psi_j \circ \gamma : [0, 1] \rightarrow \mathbf{R}^2$ are non-decreasing. We observe in particular, that each of the two pieces of $\partial W_j \cap \overline{U}$ are admissible arcs.

By Proposition 6.4, we can choose the neighborhood U of p_{fix} above so that its intersections with the wedges W_j are ‘pushed out’ by f . That is, $f(W_j \cap \partial U) \cap U = \emptyset$. This assumption and Propositions 6.2 and 6.6 imply immediately that

Corollary 6.7. *If $\gamma : [0, 1] \rightarrow W_i$ is an admissible arc, then (after reparametrizing) so is $f_{ij} \circ \gamma$.*

Theorem 6.8. *For each $j = 0, 1, 2$, the set $\bigcap_{n \in \mathbf{N}} f^{3n}(W_j) \cap U$ is an admissible arc.*

Proof. For any closed $E \subset \overline{U}$, we let $\text{Area}(E)$ denote the area with respect to the f -invariant two form η . Since U avoids the poles of η , we have $\text{Area}(U) < \infty$.

By Proposition 6.2 and our choice of U , the sets $f^{3n}(W_j) \cap \overline{U}$ decrease as n increases. In particular, Proposition 6.4 tells us that $\text{Area}(f^{3n}(W_j) \cap \overline{U}) \rightarrow 0$. Finally, for every $n \in \mathbf{N}$, $U \cap \partial f^{3n}(W_j) = f^{3n}(\partial W_j) \cap U$ consists of two admissible curves meeting at p .

Admissibility implies that in the coordinates Ψ_j , the two curves bounding $f^{3n}(W_j)$ in U are both graphs over the line $y = x$ of functions with Lipschitz constant no larger than 1. The fact that the region between these two curves decreases as n increases translates into the statement that the graphing functions are monotone in n . The uniformly bounded Lipschitz constant together with the fact that the area between the curves is tending to zero, implies further that the graphing functions converge uniformly to the same limiting function and that this limiting function also has Lipschitz constant no larger than 1. \square

Observe that by Corollary 6.5, we have just characterized the intersection of the unstable set of p_{fix} with U . Since f and f^{-1} are conjugate via the involution σ , the theorem also characterizes the stable set of p_{fix} . Our final result, stated for the stable rather than the unstable set of p_{fix} , summarizes what we have established. It is an immediate consequence of Proposition 6.4, and Corollary 4.3 and Theorem 6.8.

Corollary 6.9. *When $a = 3$ the set $W^s(p_{fix}) \cap S_0$ consists of three arcs meeting transversely at p_{fix} , and it coincides with the set of points whose forward orbits are entirely contained in S_0 . All points not in $W^s(p_{fix})$ have forward orbits tending to infinity. In particular, $K = \{p_{fix}\}$*

Theorem 1.2 is an immediate consequence.

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